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Approximating solutions to algebraic equations

Introduction

- In this topic, we will
  - Describe what will be covered in the next topic
    - Approximating solutions to algebraic equations
  - Introduce some of the terminology
  - Introduce the approaches that will be used
  - Describe the upcoming lectures

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Solutions to equations

- In elementary school,
  you learned to solve a simple linear equation $ax = b$
  - In secondary school,
    you learned how to solve quadratic equations:
    $$ax^2 + bx^2 + c = 0$$
  - However, there are simple equations that have no exact solution
    $$x^5 + x^2 + 1 = 0$$
- There are more complex equations, as well:
  $$\sin(x^2) - x^2\cos(x) + 1 = \cos(x + \sin(x)) + 5$$

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Linear and non-linear equations

- A linear equation in one or more variables is any equation that can be written in the form
  $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
- Any equation that is not a linear equation is a *non-linear* equation:
  $$\sin(x^3 + 1) + x^2 - 1 = 0$$

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Systems of equations

- In linear algebra, you learned to solve
  a system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{b}$
- You could have a system of non-linear equations

$$x^2 + 2xy + 3y = 4$$
$$x^2 - 5 = 3y + 5y^2$$

  - A solution is any pair (x, y) that satisfies both these equations
- Given a system of equations where at least one is non-linear,
  we will describe them as a system of non-linear equations
- In general, having n unknowns requires n equations
  - Each equation provides one further constraint

Root-finding problems

- Any equation can be rewritten as a root-finding problem:
  - Solving $\sin(x^5) - x^2\cos(x) + 1 = \cos(x + \sin(x)) + 5$ is equivalent to solving

$$\sin(x^5) - x^2\cos(x) + 1 - \cos(x + \sin(x)) - 5 = 0$$

- Any system of equations can be rewritten as a root-finding problem
  - Turn each equation into a root-finding problem

$$x^2 + 2xy + 3y - 4 = 0$$
$$x^2 - 5 - 3y - 5y^2 = 0$$

  - We are finding the root of a vector-valued function of a vector variable: $\mathbf{f}(\mathbf{u}) = \begin{pmatrix} u_1^2 + 2u_1u_2 + 3u_2 - 4 \\ u_1^2 - 5 - 3u_2 - 5u_2^2 \end{pmatrix}$

Root-finding problems

- Apart from linear equations and systems of linear equations, we will reformulate all equations to be a root-finding problem and thus find algorithms for:
  - Approximating a solution to a non-linear equation in one variable
  - Approximating a solution to a system of *n* non-linear equations in *n* variables
- The first we will write as $f(x) = 0$ where
  - *f* is a real- or complex-valued function of a real or complex variable, respectively
- The second we will write as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ where:
  - $\mathbf{x}$ is an *n*-dimensional vector
  - $\mathbf{f}$ is a vector-valued function of a vector variable

Approximating solutions to equations

- We will assume you know how to solve $ax = b$
- We will start by approximating solutions to a non-linear equation of one variable with:
  - Newton's method
  - The bisection method
  - The bracketed secant method
  - The secant method
- We have already discussed partial pivoting and the Jacobi method
  - The Gauss-Seidel method
  - Successive over-relaxation
- We will generalize Newton's method for approximating a solution to a system of *n* non-linear equations in *n* unknowns

Quadratic equations

- However, we will start with finding a solution to a quadratic equation

Summary

- Following this topic, you now
  - Have an overview of the ideas to be covered in this topic
  - Know how to describe equations as being linear or non-linear
  - Understand that we will use different approaches for each:
    - Matrix-vector equations for linear equations
    - Root-finding problems for non-linear equations

References

[1] https://en.wikipedia.org/wiki/System_of_linear_equations
[2] https://en.wikipedia.org/wiki/Nonlinear_system

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
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